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Note

On indexable graphs

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Abstract

Let $G = (V, E)$ be a (p, q) graph. G is said to be strongly indexable if there exists a bijection $f: V \to \{0, 1, 2, \dots, p-1\}$ such that $f^+(E) = \{1, 2, \dots, q\}$, where $f^+(uv) = f(u) + f(v)$ for any edge $uv \in E$. G is said to be indexable if f^+ is injective on E. In this paper we construct classes of strongly indexable unicyclic graphs. We also prove that trees and unicyclic graphs are indexable.

1. Introduction

By a graph we mean a finite, undirected, connected graph without loops or multiple edges. We denote by p and q the order and size of G. Terms not defined here are used in the sense of Harary [3]. The concept of indexable graph was introduced by Acharya and Hegde [1].

Definition 1.1. Let $G = (V, E)$ be a (p, q) graph. A labeling of G is a bijection $f: V \to \{0, 1, 2, \dots, p-1\}$. For any edge $e = uv$ of G, let $f^+(e) = f(u) + f(v)$. G is said to be indexable if there exists a labeling f such that $f^+: E \to N$ is injective and f is called an indexer of G. G is said to be strongly indexable if $f^+(E) = \{1, 2, \dots, q\}$ and f is called a strong indexer of G.

Acharya and Hegde [1] have conjectured that all unicyclic graphs are indexable. In this paper we prove this conjecture and also prove that all trees are indexable using breadth first search (BFS) algorithm as given in [2]. We construct several classes of strongly indexable unicyclic graphs. We need the following results.

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Theorem 1.2 (Acharya and Hegde [1]). *Every strongly indexable graph has exactly one nontrivial component which is either a star or has a triangle.*

Corollary 1.3 [Acharya and Hegde [1]). *If G is a strongly indexable graph with a triangle then any strong indexer of G must assign zero to a vertex of the triangle in G.*

Theorem 1.4 (Niven and Zuckerman [4], p. 101). *The equation $ax + by = c$, where a, b, c are integers and $a, b \neq 0$ has integer solution for x and y if and only if the g.c.d g of (a, b) divides c. Moreover, if x_0, y_0 are integers such that $ax_0 + by_0 = g$ then every integral solution x_1, y_1 of $ax + by = c$ can be written in the form $x_1 = (c/g)x_0 + (b/g)t$ and $y_1 = (c/g)y_0 - (a/g)t$, where t is any arbitrary integer.*

2. Main results

It follows from Theorem 1.2 that if a unicyclic graph is strongly indexable then the unique cycle must be a triangle. In the following theorems we give several families of unicyclic graphs which are strongly indexable.

Theorem 2.1. *Let G be a unicyclic graph with a unique triangle (u, v, w, u) such that each vertex different from u, v, w has degree one. Let m_1, m_2, m_3 be the number of pendant vertices adjacent to u, v, w respectively. Then G is strongly indexable if and only if there exist two distinct positive integers x and y such that one of the following holds:*

(i) $m_1 = x + y - 3 + m_2(x - 1) + m_3(y - 1)$,
(ii) $m_2 = x + y - 3 + m_1(x - 1) + m_3(y - 1)$,
(iii) $m_3 = x + y - 3 + m_2(x - 1) + m_1(y - 1)$.

Proof. Suppose (i) holds. Let $N(u) - \{v, w\} = \{u_1, u_2, \ldots, u_{m_1}\}$, $N(v) - \{u, w\} = \{v_1, v_2, \ldots, v_{m_2}\}$ and $N(w) - \{u, v\} = \{w_1, w_2, \ldots, w_{m_3}\}$.

Define $f: V \to N$ by $f(u) = 0$, $f(v) = x$, $f(w) = y$, $f(w_j) = x + jy$, $1 \leqslant j \leqslant m_3$, $f(v_k) = (m_3 + 1)y + kx$, $1 \leqslant k \leqslant m_2$, and assign the remaining m_1 unused labels arbitrarily to the m_1 pendant vertices $u_1, u_2, \ldots, u_{m_1}$ adjacent to u. Clearly, f is a strong indexer of G. The proof is similar if (ii) or (iii) holds.

Conversely, let f be a strong indexer of G. By Corollary 1.3 one of $f(u)$, $f(v)$ or $f(w) = 0$. Suppose $f(u) = 0$, $f(v) = x$ and $f(w) = y$. Then $f^+(v, w) = x + y$ and hence all the values less than $x + y$ other than x and y should necessarily be assigned to the pendant vertices adjacent to u and $x + y$ should be assigned to a pendant vertex adjacent to v or w. Without loss of generality, we assume that $x + y$ is assigned to a pendant vertex adjacent to w. Then all the values i, where $x + y < i < x + 2y$ should necessarily be assigned to the pendant vertices adjacent to u. Thus, whenever we assign a value to a pendant vertex adjacent to w, we should necessarily assign $y - 1$ values to the pendant vertices adjacent to u and whenever we assign a value to a pendant vertex adjacent to v, we should necessarily assign $x - 1$ values to the pendant vertices adjacent to u. Hence, $m_1 = x + y - 3 + m_2(x - 1) + m_3(y - 1)$. Similarly when $f(v) = 0$, (ii) holds and when $f(w) = 0$, (iii) holds. $\square$

Remark 2.2. Eq. (i) of Theorem 2.1 can be put in the form $ax + by = c$, where $a = 1 + m_2$, $b = 1 + m_3$ and $c = m_1 + m_2 + m_3 + 3$. Hence by Theorem 1.4, (i) has integral solutions if and only if the g.c.d. g of $1 + m_2$, $1 + m_3$ divides $1 + m_1$. By Euclidean algorithm we determine integers x_0 and y_0 such that $(1 + m_2)x_0 + (1 + m_3)y_0 = g$. Let $D_1 = \{t \in Z/(c/g)x_0 + (b/g)t > 0\}$ and $D_2 = \{t \in Z/(c/g)y_0 - (a/g)t > 0\}$. If there exists $t \in D_1 \cap D_2$ such that $(c/g)x_0 + (b/g)t \neq (c/g)y_0 - (a/g)t$, then there exist distinct positive integers x and y satisfying (i). For example if $m_1 = 21$, $m_2 = 3$ and $m_3 = 5$, then $g = 2$, $x_0 = -1$, $y_0 = 1$ and $D_1 = \{t \in Z/-16 + 3t > 0\}$ and $D_2 = \{t \in Z/16 - 2t > 0\}$. Clearly, $6 \in D_1 \cap D_2$ and $x = 2$, $y = 4$ is a solution of (i). Hence the corresponding graph G is strongly indexable. Similarly it can easily be verified that if $m_1 = 5, m_2 = 8$ and $m_3 = 6$ there does not exist positive integers x, y satisfying any of the Eqs. (i)–(iii) of Theorem 2.1 and hence the corresponding graph G is not strongly indexable.

Theorem 2.3. *Let G be a unicyclic graph consisting of a unique triangle (v_1, v_2, v_3, v_1)* with $\deg v_2 = \deg v_3 = 2$, a path $P = (v_1, u_1, u_2, \ldots, u_n)$ *of length n and k pendant vertices adjacent to v_1. Then G is strongly indexable if and only if $k = a_n$,* where $a_2 = x + y + z - 5$, $a_3 = 2x + 2y + z - 6$, $a_n = a_{n-1} + a_{n-2} + n$ *for $n > 3$ and x, y, z are three distinct positive integers with $z \neq x + y$.*

Proof. Let $\{w_1, w_2, \ldots, w_k\}$ be the pendant vertices adjacent to v_1. Define $f: V \to N$ by $f(v_1) = 0$, $f(v_2) = x$, $f(v_3) = y$, $f(u_1) = z$, $f(u_2) = x + y$, $f(u_i) = f(u_{i-1}) + f(u_{i-2})$, $3 \leqslant i \leqslant n$, and assign the remaining labels arbitrarily to $w_1, w_2, w_3, \ldots, w_k$. Clearly, f is a strong indexer of G.

Conversely, let f be a strong indexer of G. By Corollary 1.3, either $f(v_1) = 0$ or $f(v_2) = 0$ or $f(v_3) = 0$. Suppose $f(v_2) = 0$. Then $\{f(v_1), f(v_3)\} = \{1, 2\}$. If $f(v_3) = 1$ and $f(v_1) = 2$ then there is no edge with label 4. If $f(v_3) = 2$ and $f(v_1) = 1$ then the vertices $u_1, w_1, w_2, \ldots, w_k$ must be given the labels $3, 4, \ldots, k + 3$ in some order. Hence, $f(u_2) \geqslant k + 4$ and there is no edge with label $k + 5$. Thus $f(v_2) \neq 0$. Similarly $f(v_3) \neq 0$ and hence $f(v_1) = 0$. Let $f(v_2) = x$, $f(v_3) = y$ and $f(u_1) = z$. Then $f(u_2) = x + y$ since otherwise there is no edge with label $f(u_2)$. Suppose $n = 2$, then $f^+(u_1u_2) = x + y + z$ and hence any value i with $1 \leqslant i < x + y + z$, $i \neq x, y, z, x + y$ must be assigned to a pendant vertex adjacent to v_1 so that $a_2 = x + y + z - 5$. Similarly, where $n = 3$, $f(u_3) = x + y + z$ and $f^+(u_2u_3) = 2x + 2y + z$ so that any value j with $1 \leqslant j < 2x + 2y + z$, $j \neq x, y, z, x + y, x + y + z$ must be assigned to a pendant vertex adjacent to v_1 giving $a_3 = 2x + 2y + z - 6$. Suppose $n > 3$. Then $f(u_i) = f(u_{i-1}) + f(u_{i-2})$ for all $i > 3$ and it can be proved by induction on n that $f^+(u_{n-1}u_n) = a_n + n + 3$, for all $n > 3$. Hence, $q = a_n + n + 3$ so that $k = a_n$. $\square$

Remark 2.4. Given n and k, the equation $k = a_n$ of Theorem 2.3 can be put in the form $a(x + y) + bz = k + n + 3$, where a and b are relatively prime. By Euclidean algorithm we determine integers x_0 and y_0 such that $ax_0 + by_0 = 1$. Let $D_1 = \{t \in Z \mid cx_0 + bt > 2\}$ and $D_2 = \{t \in Z \mid cy_0 - at > 0\}$. If there exists $t \in D_1 \cap D_2$ such that $cx_0 + bt \neq cy_0 - at$, then there exist three distinct positive integers x, y and z with $z \neq x + y$ satisfying the equation $k = a_n$. For example, if $k = 11$ and $n = 6$ there does not exist distinct positive integers x, y, z with $z \neq x + y$ and satisfying the equation $k = a_n$ so that the corresponding graph G is not strongly indexable. If $n = 6$ and $k = 124$ then $x = 1$, $y = 5$ and $z = 17$ is a solution of $k = a_n$ and hence the corresponding graph G is strongly indexable.

Theorem 2.5. *All trees are indexable.*

Proof. Let T be a tree and let v be any vertex of T. Starting from v, we visit all the vertices of T using BFS and label the vertices with the consecutive numbers $0, 1, 2, \dots, p-1$ in the order in which they are visited. Let e_1 and e_2 be any two edges of T. If e_1 and e_2 are adjacent then obviously $f^+(e_1) \neq f^+(e_2)$. Suppose e_1 and e_2 are non-adjacent. Let $e_1 = u_1u_2$, $e_2 = u_3u_4$ and let $f(u_1) < f(u_2), f(u_3), f(u_4)$. Now since T is a tree at least one of the vertices u_3, u_4 is not adjacent to u_1. Suppose u_4 is not adjacent to u_1. Then, $f(u_4) > f(u_2)$ and hence it follows that $f^+(e_1) \neq f^+(e_2)$. □

Theorem 2.6. *All unicyclic graphs are indexable.*

Proof. Let G be a unicyclic graph with unique cycle $C_n = (u_1, u_2, \dots, u_n, u_1)$. When $G = C_n$, starting from u_1, we visit all the vertices of G using BFS and label the vertices with the consecutive integers $0, 1, 2, \dots, n-1$ in the order in which they are visited. Clearly this labeling is an indexer of G. If $G \neq C_n$ choose any vertex u_1 of C_n such that $\deg u_1 \geqslant 3$.

Case (i): *n is even.* Starting from u_1 we visit all the vertices of G using BFS subject to the following restrictions. We first visit all the neighbours of u_1 not on the cycle so that u_n and u_2 are visited consecutively. Without loss of generality, assume that u_n is visited first. In general, if $0 \leqslant i \leqslant n/2$ while we visit the neighbours of u_{n-i}, we first visit the neighbours that are not on the cycle and for neighbours of u_{i+2}, we first visit its neighbour on the cycle. Label the vertices of G with the consecutive numbers $0, 1, 2, \dots, p-1$ in the order in which we visit them. Clearly, $f(V) = \{0, 1, 2, \dots, p-1\}$. Since G is a unicyclic graph, BFS algorithm gives exactly one back edge u_ku_{k+1}, where $k = n/2$. Now let e_1 and e_2 be any two edges of G. If either e_1 and e_2 are adjacent or $e_1, e_2 \neq u_ku_{k+1}$, then obviously $f^+(e_1) \neq f^+(e_2)$. Suppose $e_1 = u_ku_{k+1}$, e_1 and e_2 are non-adjacent and $e_2 = w_1w_2$ with $f(w_1) < f(w_2)$.

If $f(w_2) < f(u_k)$ then $f(w_1) < f(w_2) < f(u_k) < f(u_{k+1})$. If $f(u_k) < f(w_2) < f(u_{k+1})$ then $f(w_1) < f(u_{k+2}) < f(u_k) < f(w_2) < f(u_{k+1})$. If $f(u_k), f(u_{k+1}) < f(w_2)$, then $f(w_1) > f(u_k)$, so that $f(u_k) < f(w_1) < f(u_{k+1}) < f(w_2)$ or $f(u_k) < f(u_{k+1}) < f(w_1) < f(w_2)$. In all cases we have $f^+(e_1) \neq f^+(e_2)$ so that f is an indexer of G.

Case (ii): *n is odd.* Starting from u_1 we visit all the neighbours of u_1, using BFS such that we first visit u_n, then the neighbours of u_1 that are not on the cycle and then u_2. Next visit u_{n-1} and then the other neighbours of u_n so that u_2 and u_{n-1} are visited consecutively. Now proceeding as in case (i) we obtain an indexer of G. □

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